IITM Journal of Management and IT
Vol. No. 15 Issue No. 1
June 2024

# Computational Development of Duality for Mathematical Programming Problem: Historic Analysis

MEGHA SHARMA*

**The theory of classical duality programming problem launched in 1948, an alternative approach to solve the mathematical programming problem. The questions of whether the duality can be seen as a multiple discovery and why the duality results were served as boon in solving mathematical programming problems and employed in the different fields. On the basis of a contextualized historic analysis of this work, the significance of contexts both computationally and theoretically for these questions are illustrated which includes the role played in mathematical programming problem.**

**Keywords:** *Nonlinear programming problem, Duality, Machine Learning*

## 0. INTRODUCTION

Mathematical Programming problems are inherently characterized by the multiple, conflicting and incommensurate aspects of evaluation for the merits of alternative solutions. The axes of evaluation are based on constrained objective function maximize (or minimize) in the framework of decision alternatives.

Optimality results are derived to explore the solution of mathematical programming rooted by Edge worth and Pareto. Soon after that rapid advances indecision sciences that few remember the contribution of great pioneers that started it all. Among them Hitch-Koopmans [16, 21] were mathematician and economist. Koopmans was also awarded by Noble prize in 1975 in Economics for his work on the theory of optimum allocation of resources. They designed transportation problem during the era of World War II. They studied the following problem:

* Associate Professor, Department of Management,Institute of Information Technology & Management, GGSIPU, drmsmath22@gmail.com, megha_sharma @iitmipu.ac.in

" Let $y = \sum a_{ij} x_{ij}$ for $i = 1, 2, \ldots\ldots m$; $j = 1, 2, \ldots\ldots n$; $a_{ij}$ be the cost of product supply from *i*th factory to *j*th city and $x_{ij}$ represents number of tons shipped Then aim is to evaluate distribution at minimum cost [16]." In 1947, Simplex Method had been discovered by Dantzig [6] and further extension of linear programming studied by several researchers [7, 37]. Table 1 entails the contextualized work behind the foundation of duality programming problem:

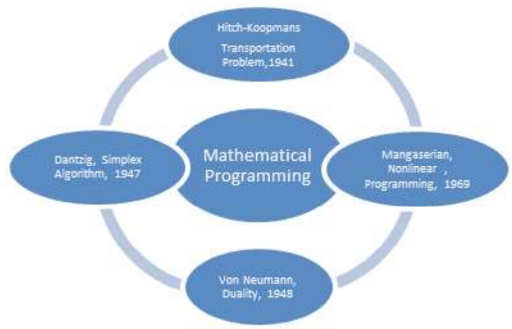


**Table 1: Historical Analysis**

The idea behind this study is to make coherent set that focus on advancing the theory, algorithms and applications of optimization. Duality concept began in 1948 for linear programming introduced by John Von Neumann. Duality plays quintessential role in finding the solution efficiently. Consequently, Gale and Kuhn-Tucker 1951 [13, 14, 15] derived the results based on Ferkas lemma [10-11] which played crucial role in the development of duality. In 1948, Fritz John conditions obtained for nonlinear programming [18]. Duality in optimization problems is concerned with the respective primal for the determination of minimum or maximum of a objective which are require to satisfy given constraints. Duality in scalar nonlinear programming originated by Dorn [7]. A dual programming problem is formulated for nonlinear programming problem by Wolfe [35] and proved duality results under the assumption of convexity. Duality is reflected in economic applications to show the reciprocity conditions between production and consumption and also between production and profit. Duality is also played important role in Support Vector Machine investigations, transportation problems etc.

Many authors have been studied the duality results for the certain optimization problems. This act as a technique to addresses the intrinsic uncertainty in models of real-world problems. Duality in mathematical programming problems is one of the approach to tackles mathematical programming models which possesses some interesting characteristics. Mangasarain [25-26] derived the duality results for nonlinear programming problems. It has been found that duality is significant due to computational advantage as it provides tighter bound for objective function. Further, Bhatia [1], Chandra [2], Craven [3-4], Egudo [8], Jahn [17], Khurana [19], Klinger [20], Lalitha et al [22], Singh et al [36], Suneja et al [29-30], Weir et al [34] etc derived duality results and extend the notion of duality.

The historical facts highlights the following questions:

Why were the reactions of the researchers so different in this case?

Why did results based on the duality have such an enormous impact in nonlinear programming?

This paper is framed on these questions: They will be addressed and discussed on the basis of contextualized historical analysis of the computational development of duality. Both scientific and social context will be considered and the paper will end on the discussion on the role played in Support vector machine have been discussed.

## 1. PRELIMINARIES AND NOTATIONS

Let $M$ be nonempty subset of Euclidean dimensional space of $n$. The following conventions have been used for vectors $x \in$ ”!$^n$

$x \leqq y \Leftrightarrow x_i \leqq y_i,\ i = 1, 2, \ldots n;$

$x \leq y \Leftrightarrow x \leqq y,\ x\ ^1\ y;$

$x < y \Leftrightarrow x_i < y_i,\ i = 1, 2, \ldots n;$

$x \nleqq y$ is the negation of $x \leqq y$;

$x \nless y$ is the negation of $x < y$.

For $x, y$ “”!, we use the usual notation $\leq$ .

If we select any two point in arbitrarily in $n$-dimensional finite space and all points of line joining these two point belong to set then point of set is convex. This had been evaluated that minimum value of a linear function on a convex point set is unique [21]. Further, various generalizations of convex functions in this context have been appeared in literature which act as backbone in the study of duality programming. Convexity act as important aspect in economics, management sciences, and applied optimization theory and defined over convex sets [1]. Several generalizations of convex functions have been studied in the literature by various authors. Reiland [27], Rockafellar [28], Suneja et al. [29-30], Yen and Sach [38], Weir [33] and Weir et al [34].

**Definition 0.1** The set $M$ is said to be convex if

$(1-\theta)\bar{x} + \theta x \in M$, for all

$\bar{x},\ x \in M,\ 0 \leq \theta \leq 1$. Let $M$ be nonempty subset of ”!$^n$, $f$: $M$ ’!”! be a real-valued function and differentiable function and $\nabla f = \left( \frac{\partial f}{x_1}, \frac{\partial f}{x_2}, ..... \frac{\partial f}{x_n} \right)$.

**Definition 0.2** The function *f* is said to be convex at $\bar{x} \in M$ if *S* is convex and for every $x \in M$,

$$f\left(\left(1-\theta\right)\bar{x}+\theta x\right) \leqq \left(1-\theta\right) f\left(\bar{x}\right)+\theta f\left(x\right),$$
$$0 \leq \theta \leq 1.$$

## 2. DUALITY PROGRAMMING APPROACH

Linear programming becomes the standard technique for improving efficiency in the different fields of management sciences. It is being used in solving many problems finding maximum profit, minimum cost, minimum overtime, maximum production and other similar problems and improving economic productivity.

Let us consider, without loss of generality, the following linear programming problem

(LP) Max (or Min) $\sum_{j=1}^{n} c_j x_j$

St $\sum_{j=1}^{n} a_{ij} x_j \leqq$ (or $\geqq$) $b_j$; $i = 1, 2,...m$

*Or* $Ax \leqq$ (or $\geqq$ ) b

$x_j$ 0; j =1, 2......n

The dual programming problem associate (LP) problem as follows

*(PD) Min*

*St* $A'y$ ; j = 1, 2,......*n*

This has been quantified that linear program problem coefficient matrix expresses the consumption of physical quantity of inputs necessary to produce the set of quantities of outputs. In the dual space it expresses the creation of the economic values associated with the outputs from the input sets.

## 3. HISTORICAL ANALYSIS OF DUALITY IN NONLINEAR PROGRAMMING

Decision makers encounters several situations in which they found that real-world cannot always be linear in nature. The mathematical programming for such problems is termed as nonlinear programming problem. The nonlinear programming problem is finite dimensional optimization problem where the variables have to fulfil some inequality constraints. An *n*-dimensional vector *x* in ”!$^n$, which satisfy the constraint is said to be feasible:

(Primal) Min *f(x)*

St $g(x)$ 0;

Where $f$ and $g_j$; $j$ =1, 2....*m* are real-valued nonlinear differentiable functions

**Kuhn-Tucker Theorem** [22] Suppose *M* is a nonempty open set of ”!$^n$ and $x^*$ is feasible and Ñ$g_i(x^*)$ for the binding –or- active constraints ie the constraint $g_i$ for which $g_i(x^*) = 0$ are linearly independent then Kuhn-Tucker necessary conditions hold:

$f(x^*)$ is minimum for nonlinear programming model above are that there exists scalar multipliers such that

In 1961, Philip Wolfe [35] worked on the nonlinear programming problem and designed new programming problem termed as Dual programming problem which reduces to classical dual problems studied by Dorn [7] and investigated the duality results based on the qualification which serves to rule out the certain singularities which might otherwise occur on the boundary of the constraint set. The constraint set has an interior point relative to convexity determined by those constraints which are nonlinear.

The dual has been designed by Gale [13-14] and Kuhn-Tucker [22] and Goldman and Tucker [15] as follow

(Primal) Min *f(x)*

St $g(x)$ 0;

where $f$ and $g_j$; $j$ =1, 2....*m* are concave real-valued nonlinear differentiable functions:

(Dual) Max

St

*u “M ë “* ”!$^m$, *ë 0*

The relation between the solution of nonlinear programming problem and its dual has been discovered based on the classical *Principal of Duality* for nonlinear programming problem.

**Principal of Duality [35]** If $x^*$ solves the primal problem then $(x^*, u^*)$ solves the dual

If

∇ Primal has solution Dual is consistent

∇ Primal is bounded and has no solution Dual is inconsistent

∇ Primal has unbounded solution Dual is inconsistent

∇ Primal is Inconsistent Dual is unbounded and inconsistent

After that decision makers studied generalized aspects of duality [12] and derived the optimality and duality results for several optimization problems

| Duality | Author | Year | Reference |
|---|---|---|---|
| Generalized Duality for Nonlinear Programming | Craven | 1980 | [5] |
| Second-Order Duality for Vector Optimization | Mangasarin | 1969 | [25-26] |
| Symmetric Duality | Chandra et al | 2003 | [2] |

Mangasarian [25-26] worked on duality programming model and constructs the various duality results based on the above problems under the assumption of convexity. The duality results are derived for continuous programming for certain problems

## 4. COMPUTATIONAL DEVELOPMENT OF DUALITY: SVM

Duality programming problem has been applied in Machine learning approach to resolve the multiple problems. Machine learning is a collective data analysis tools which adopt iterative approaches to derived optimal prediction to enhance the efficiency parameter which analysed the smart alternative to be chosen from the complex data without being explicitly programmed. The dual optimization technique had been used by several researchers [31-32] to solve the SVM problem. The SVM reduces in primal form [9] as follows

Primal

for all *i*

; where is the slack variable to find error at $(x_i, y_i)$, C is regularization parameter and is reproducing kernel Hilbert Space and $h(x) = wx + b$ is subset of the hyperplanes

The iterative approaches have been used to solve these problems and its dual formulation is as follows [9]:

Dual

St for all *i*

Kernel is defines as

There is a unique optimal solution for each SVM parameters

## 5. CONCLUSION

The duality results shows that a mathematical theorem in itself its pure mathematical content can also play an essential role in social context also. Even though the these results are generalized in certain vector optimization problems. The significance of results and its potential for stimulating further research in subsequent areas are determined in mathematical and computational context within which it was discovered. The duality programming model was an important formulation in mathematical discipline in which Wolfe [35] and Mond & Weir [24] were working. The discipline also received the huge exposure to its applications illustrated in find the optimal solution in Support vector machine with large data sets.

REFERENCES

1. BHATIA (M.) Some Aspects of Optimality and Duality in Vector Optimization; University of Delhi, 2012.
2. CHANDRA (S.), CRAVEN (B.D.) and MOND (B.). Symmetric Duality in Fractional Pogramming Zeitchrift fr Operations Research, 29, 59-64, 1985.
3. CRAVEN (B. D.), Invex Functions and Constrained Local Minima, Bulletin of Australian Mathematical Society, 24 (3), 357-367, 1981.
4. CRAVEN (B. D.), Strong Vector Minimization and Duality, Zamm-zeitschrift Fur Angewandte Mathematik Und Mechnik., 60, 1980.
5. COTTLE (R. W.), A theorem of Fritz John in Mathematical Programming, RAND Memorandum RM-3858-PR, Traces and Emergence of Nonlinear Programming, October, 111-123, 1963.
6. DANTZIG (G. B.), Linear Programming and Extensions, Princeton University Press, Princeton, New Jersey 1963.
7. DOM (W. S.), Duality in Quadratic Programming, Quarterly Journal of Applied Mathematics, 18, 155-162, 10.1090/qam/112751, 1960.
8. EGUDO (R. R.), Multiobjective Fractional Duality, Bulletian Australian Mathematical Society, 37, 367-378, 1988.
9. EVGENIOU (T.) and PONTIL (M.), Support Vector Machines: Theory and Applications, 249-257, 2001.
10. FARKAS (G. J.). Die algebraische Grundlage der Anwendungen des mechanischen Princips von Fourier. Mathematische und Naturwissenschaftliche Berichte aus Ungarn 16, 154-157, 1899.
11. FARKAS (G. J.). Theorie der einfachen Ungleichungen. Journal fr die reine und angewandte Mathematik, 124, 1-27, 1901.
12. FIGUEROA-GARCIA (J. C.), HEMANDAZ (G.), FRANCO (C.), A Review on History, Trends and Prespective of Fuzzy Linear Programming, Operations Research Perspective,https://doi.org/10.1016/j.orp.2022.100247, 2022.
13. GALE (D.), Convex Polyhedral Cones and Linear Inequalities, T. C. Koopman's (Eds) Actively Analysis of Production and Allocations, John Willey and Sons, Inc. New-York, 1951.

14. GALE (D.), Neighbouring Vertices on a convex Polyhedron, in Annals of Mathematics Studies 38, New Jersey, Princeton University Press, 253-263, 1956.
15. GOLDMAN (A. J.) and TUCKER (A.W.), Polyhedral Convex Cones in Annals of Mathematic Studies, Princeton University Press, 38, 19-39, 1956.
16. HITCHCOCK (F. L.), The Distribution of a Product from Several Sources to Numerous Localities, MIT Journal of Mathematics and Physics 20:224–230, 1941.
17. JAHN (J). Vector Optimization Theory, Applications and Extensions, Springer, Berlin, 2004.
18. JOHN (F.), "Extremum Problems with Inequalities as Subsidiary Conditions", *Studies and Essays, Courant Anniversary Volume*, 197-215, 1948.
19. KHURANA (S.), Symmetric Duality in Multiobjective Programming Involving Generalized Cone-Invex Functions, European Journal of Operational Research, 165 (3), 592–597, 2005.
20. KLINGER (A), Improper Solutions of Vector Minimum Problems, Operations Research, 15, 570-572, 1967.
21. KOOPMANS (T. C.), Optimum utilization of the transportation system, Proceeding of the International Statistical Conference, Washington D.C., 1947
22. KUHN (H. W.) and TUCKER (A. W.), Nonlinear Programming Second Berkely Symposium on Mathematical Statistics and Probability, University of California, Press California, 481-493, 1951.
23. LALITHA (C. S.), SUNEJA (S. K.) and KHURANA (S.), Symmetric Duality Involving Invexity in Multiobjective Fractional Programming, Asia-Pacific Journal of Operations Research, 20, 57-72, 2003.
24. MOND (B.), WEIR (T.), Generalized Concavity in Optimization and Duality, In: Schaible S and Ziemba W. T. (Eds), Generalized Concavity in Optimization and Economics, New-York, 263-279, 1981.
25. MANGSANN (O. L.), Second and Higher-Order Duality in nonlinear Programming, Journal of Mathematical Analysis and Applications, 51, 607-620, 1975.
26. MANGSANN (O. L.), Nonlinear Programming, McGraw-Hill, New York, 1969.
27. REILAND (T. W.), Nonsmooth Invexity, Bulletin of Australian Mathematical Society, 42 (3), 437-446, 1990.
28. ROCKAFELLAR (R. T.), Convex Functions and Duality in Optimization Problems and Dynamics, In: Kuhn, HW, Szeg GP (eds) Mathematical Systems Theory and Economics I/II, Lecture Notes in Operations

Research and Mathematical Economics, Vol 11(12), 117-141 http//doi.org/10.1007/978-3-642-46196-5-7, 1969.

29. SUNEJA (S.K.) and GUPTA (S.), Duality in Multiobjective Nonlinear Programming Involving Semilocally Convex and Related Functions, European Journal of Operational Research, 107 (3), 675–685. 1998.
30. SUNEJA (S.K.) and GUPTA (S.), and VANI Optimality and Duality in Multiobjective Nonlinear Programming Involving -Semilocally Preinvex and Related Functions, Opsearch, 44(1), 27–40, 2007.
31. TRAFALIS (T. B.), Primal-dual Optimization Methods in Neural Networks and Support Vector Machine Training, ACA https://api.semanticscholar.org/CorpusID:8936459, 1999.
32. VAPNIK (V.), Statistical Learning Theory, New York, http://api.semanticscholar. org/CorpusID:61112307, 1998.
33. WEIR (T). A Duality Theorem for Multiple Objective Fractional Optimization Problem, Bulletin of Australian Mathematical Society, 34, 415-425, 1986.
34. WEIR (T.), MOND (B.) and CRAVEN (B. D.). Weak Minimization and Duality, Numerical Functional analysis and Optimization, 9, 181–192, https://doi.org/10.1080/01630568708816230, 1987.
35. WOLFE (P.), A duality theorem for nonlinear programming, Quarterly Journal of Applied Mathematics, 19, 239-244, 1961.
36. SINGH (C); SUNEJA (SK) and RUEDA (NG). Preinvexity in Multiobjective Fractional Programming, Journal of Information and Optimization Science, 8, 293-302, 1992.
37. NEUMANN (J. V.), Discussion of Maximization Problem, Institute for Advanced Study, Princeton, New Jersey, 1947
38. YEN (N.D.) and SACH (P.H.), On Locally Lipschitz Vector Valued Invex Functions, Bulletin of Australian Mathematical Society, 47, 259–271, 1993.